DIFFERENTIAL EQUATIONS

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Differential Equations

⇨ An equation which involves unknown function and its derivatives

- ordinary differential equation (ode) : not involve partial derivatives
- partial differential equation (pde) : involves partial derivatives
- ***order*** of the differential equation is the order of the highest derivatives

Examples:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x \sin y$$

→ second order ordinary differential equation

$$\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial x} = \frac{x+t}{x-t}$$

→ first order partial differential equation

Modeling via Differential Equations

Note that the set of equations is called a **Model for the system.**

How do we build a Model?

The basic steps in building a model are:

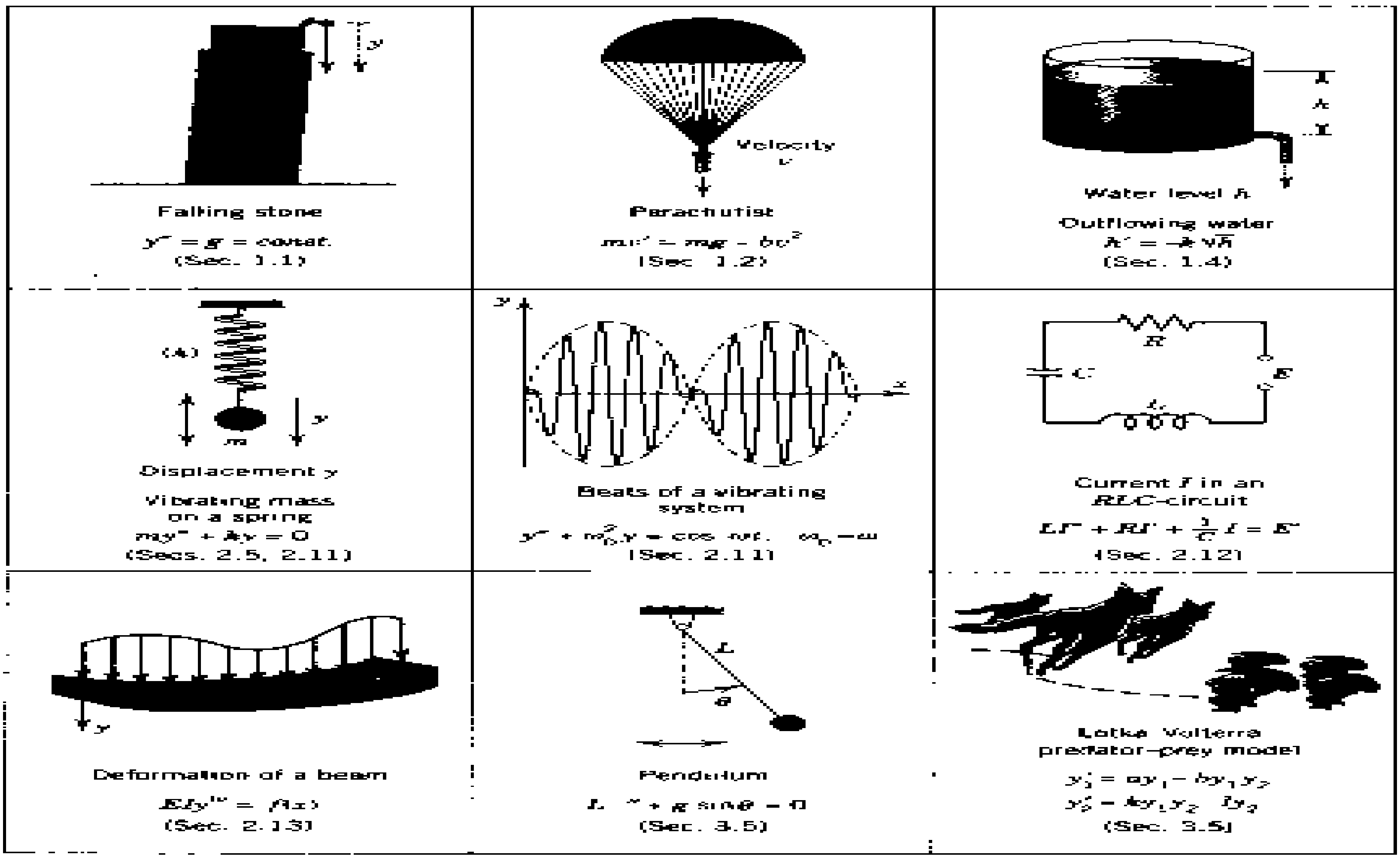
Step 1: Clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2: Completely describe the parameters and variables to be used in the model.

Step 3: Use the assumptions (from Step 1) to derive mathematical equations relating the parameters and variables (from Step 2).

Differential Equations

Some Application of Differential Equation in Engineering



Differential Equations

Larutan
Air-Garam

Konsentrasi garam : b gram/liter
Laju alir : e liter/min

Awal:
Vol. air : Vo liter
Massa garam :
a gram

Tercampur sempurna
laju alir : f liter/min

Massa garam setiap saat (menit) ?

Differential Equations

⇨ A linear differential equation of order n is a differential equation written in the following form:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$$

where $a_n(x)$ is not the zero function

⇨ **General solution** : looking for the unknown function of a differential equation

⇨ **Particular solution (Initial Value Problem)** : looking for the unknown function of a differential equation where the values of the unknown function and its derivatives at some point are known

⇨ Issues in finding solution : **existence** and **uniqueness**

1^{st} Order DE - Separable Equations

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is separable if the equation can be written in the form:

$$\boxed{f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0}$$

Solution :

1. Multiply the equation by integrating factor: $\boxed{\dfrac{1}{f_2(x)g_1(y)}}$
2. The variable are separated :

$$\boxed{\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0}$$

3. Integrating to find the solution:

$$\boxed{\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = C}$$

1^{st} Order DE - Separable Equations

Examples:

1. Solve : $4x\,dy - y\,dx = x^2 dy$

   Answer:

1^{st} Order DE - Separable Equations

Examples:

1. Solve : $$\frac{dy}{dx} = \frac{x^2 + 2}{y}$$

   Answer:

1^{st} Order DE - Separable Equations

Examples:

2. Find the particular solution of : $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x}$; $y(1) = 2$

Answer:

1^{st} Order DE - Homogeneous Equations

Homogeneous Function

f (x,y) is called homogenous of degree *n* if : $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

Examples:

$f(x,y) = x^4 - x^3 y$ → homogeneous of degree 4

$$f(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3 (\lambda y)$$
$$= \lambda^4 \left(x^4 - x^3 y\right) = \lambda^4 f(x,y)$$

$f(x,y) = x^2 + \sin x \cos y$ → non-homogeneous

$$f(\lambda x, \lambda y) = (\lambda x)^2 + \sin(\lambda x)\cos(\lambda y)$$
$$= \lambda^2 x^2 + \sin(\lambda x)\cos(\lambda y)$$
$$\neq \lambda^n f(x,y)$$

1^{st} Order DE - Homogeneous Equations

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree

Solution :

1. Use the transformation to : $\boxed{y = vx \quad \Rightarrow \quad dy = v\,dx + x\,dv}$
2. The equation become separable equation:

$$\boxed{P(x,v)dx + Q(x,v)dv = 0}$$

3. Use solution method for separable equation

$$\boxed{\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(v)}{g_1(v)}dv = C}$$

4. After integrating, v is replaced by y/x

1st Order DE – Homogeneous Equations

Examples:

1. Solve : $\left(x^3 + y^3\right)dx - 3xy^2\ dy = 0$

   Answer:

1^{st} Order DE - Homogeneous Equations

Examples:

2. Solve : $$\frac{dy}{dx} = \frac{-2x+5y}{2x+y}$$

Answer:

1^{st} Order DE – Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if : $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

The solutions are given by the implicit equation $\boxed{F(x,y) = C}$
where : $\partial F/\partial x = M(x,y)$ and $\partial F/\partial y = N(x,y)$

Solution :

1. Integrate either $M(x,y)$ with respect to x or $N(x,y)$ to y.
   Assume integrating $M(x,y)$, then :

$$F(x,y) = \int M(x,y)dx + \theta(y)$$

2. Now : $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx\right] + \theta'(y) = N(x,y)$$

or : $$\theta'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)dx\right]$$

1^{st} Order DE – Exact Equation

3. Integrate $\theta'(y)$ to get $\theta(y)$ and write down the result $F(x,y) = C$

Examples:

1. Solve : $\left(2x^3 + 3y\right)dx + (3x + y - 1)dy = 0$

   Answer:

1^{st} Order DE – Exact Equation

Examples:

2. Solve : $4xy + 1 + \left(2x^2 + \cos y\right)\dfrac{dy}{dx} = 0$

Answer:

1^{st} Order DE – Non Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is a non exact equation if : $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

The solutions are given by using integrating factor to change the equation into exact equation

Solution :

1. Check if : $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \Rightarrow function\ of\ x\ only$$

then integrating factor is $e^{\int f(x)dx}$

or if : $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \Rightarrow function\ of\ y\ only$$

then integrating factor is $e^{\int g(y)dy}$

1st Order DE – Non Exact Equation

2. Multiply the differential equation with integrating factor which result an exact differential equation
3. Solve the equation using procedure for an exact equation

1^{st} Order DE – Non Exact Equation

Examples:

1. Solve : $\left(x^2 + y^2 + x\right)dx + xy\,dy = 0$

   Answer:

1^{st} Order DE – Non Exact Equation

Examples:

2. Solve : $\dfrac{dy}{dx} = -\dfrac{3xy + y^2}{x^2 + xy}$

   Answer:

1^{st} Order DE – Linear Equation

A **first order linear differential equation** has the following general form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Solution :

1. Find the integrating factor: $\boxed{u(x) = e^{\int p(x)dx}}$

2. Evaluate : $\boxed{\int u(x)q(x)dx}$

3. Find the solution:

$$\boxed{y = \frac{\int u(x)q(x)dx + C}{u(x)}}$$

1^{st} Order DE – Linear Equation

Examples:

1. Solve : $\dfrac{dy}{dx} + 2xy = 4x$

   Answer:

1^{st} Order DE – Linear Equation

Examples:

2. Find the particular solution of :

$$y' + \tan(x)\,y = \cos^2(x), \qquad y(0) = 2$$

Answer:

2^{nd} Order DE – Linear Equation

A **second order** differential equation is an equation involving the unknown function *y*, its derivatives *y*' and *y*'', and the variable *x*. We will consider explicit differential equations of the form :

$$\frac{d^2y}{dx^2} = f(y, y', x)$$

A **linear** second order differential equations is written as:

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

When $d(x) = 0$, the equation is called **homogeneous**, otherwise it is called **nonhomogeneous**

2^{nd} Order DE – Linear Equation

To a nonhomogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \qquad (NH)$$

we associate the so called **associated homogeneous equation**

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (H)$$

Main result:

The general solution to the equation (NH) is given by:

$$\boxed{y = y_h + y_p}$$

where:

(i) y_h is the general solution to the associated homogeneous equation (*H*);

(ii) y_p is a particular solution to the equation (*NH*).

2^{nd} Order DE – Linear Equation

Basic property

Consider the homogeneous second order linear equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

or the explicit one

$$y'' + p(x)y' + q(x)y = 0$$

Property:

If y_1 and y_2 are two solutions, then:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution for any arbitrary constants c_1 ,c_2

2^{nd} Order DE – Reduction of Order

Reduction of Order Technique

This technique is very important since it helps one to find **a second solution** independent from **a known one**.

Let y_1 be a non-zero solution of: $y'' + p(x)y' + q(x)y = 0$

Then, a second solution y_2 independent of y_1 can be found as:

$$y_2(x) = y_1(x)v(x)$$

Where:

$$v(x) = \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} dx$$

The general solution is then given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

2^{nd} Order DE – Reduction of Order

Examples:

1. Find the general solution to the Legendre equation

$$\left(1-x^2\right)y''-2xy'+2y=0$$

Using the fact that : $y_1 = x$ is a solution.

2^{nd} Order DE – Homogeneous LE with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients

A second order homogeneous equation with constant coefficients is written as: $ay'' + by' + cy = 0 \quad (a \neq 0)$

where *a*, *b* and *c* are constant

The steps to follow in order to find the general solution is as follows:

(1) *Write down the* ***characteristic equation***

$$a\lambda^2 + b\lambda + c = 0 \qquad (a \neq 0)$$

This is a quadratic. Let λ_1 and λ_2 be its roots we have

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2^{nd} Order DE – Homogeneous LE with Constant Coefficients

(2) *If λ_1 and λ_2 are distinct real numbers (if **b2 - 4ac > 0**), then the general solution is:*

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

(3) *If $\lambda_1 = \lambda_2$ (if **b2 - 4ac = 0**), then the general solution is:*

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}}$$

(4) *If λ_1 and λ_2 are complex numbers (if **b2 - 4ac < 0**), then the general solution is:*

$$\boxed{y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)}$$

Where:

$$\alpha = \frac{-b}{2a} \text{ and } \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

2^{nd} Order DE – Homogeneous LE with Constant Coefficients

1. Find the solution to the Initial Value Problem

$$y'' + 2y' + 2y = 0 \quad ; \; y(\pi/4) = 2 \; ; \; y'(\pi/4) = -2$$

2^{nd} Order DE – Non Homogeneous LE

To a nonhomogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \qquad (NH)$$

we associate the so called **associated homogeneous equation**

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (H)$$

Main result:

The general solution to the equation (NH) is given by:

$$\boxed{y = y_h + y_p}$$

where:

(i) y_h is the general solution to the associated homogeneous equation (*H*);

(ii) y_p is a particular solution to the equation (*NH*).

2nd Order DE – Non Homogeneous LE Method Undetermined Coefficients

We will guess the form of y_p and then plug it in the equation to find it.

However, it works only under the following two conditions:

- ❑ the associated homogeneous equations has constant coefficients
- ❑ the nonhomogeneous term $d(x)$ is a special form

$$\boxed{d(x) = P_n(x)e^{\alpha x}\cos(\beta x) \quad \text{or} \quad d(x) = L_n(x)e^{\alpha x}\sin(\beta x)}$$

where $P_n(x)$ and $L_n(x)$ are polynomial functions of degree n

Note: we may assume that $d(x)$ is a sum of such functions

Then a particular solution yp is given by:

$$\boxed{y_p(x) = x^s\left(T_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right)}$$

Where:

$$T_n(x) = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n$$
$$R_n(x) = B_0 + B_1x + B_2x^2 + \ldots + B_nx^n$$

2nd Order DE – Non Homogeneous LE Method Undetermined Coefficients

The steps to follow in applying this method:

1. Check that the two conditions are satisfied
2. Write down the characteristic equation and find its root
$$a\lambda^2 + b\lambda + c = 0$$
3. Write down the number $\alpha + \beta i$
4. Compare this number to the roots of the characteristic equation

If : $\alpha + \beta i$ is not one of the roots $\Rightarrow s = 0$
$\alpha + \beta i$ is one of the distincroots $\Rightarrow s = 1$
$\alpha + \beta i$ is equal to both roots $\Rightarrow s = 2$

5. Write down the form of particular solution
$$y_p(x) = x^s\left(T_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right)$$
Where: $T_n(x) = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n$ $R_n(x) = B_0 + B_1x + B_2x^2 + \ldots + B_nx^n$
6. Find constant *A* and *B* by plugging y_p solution to original equation

2nd Order DE – Non Homogeneous LE Method Undetermined Coefficients

1. Find a particular solution to the equation

$$y'' - 3y' - 4y = 2\sin(x)$$

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

If the nonhomogeneous term $d(x)$ consist of several terms:

$$d(x) = d_1(x) + d_2(x) + \ldots + d_N(x) = \sum_{i=1}^{i=N} d_i(x)$$

We split the original equation into N equations

$$a(x)y'' + b(x)y' + c(x)y = d_1(x)$$
$$a(x)y'' + b(x)y' + c(x)y = d_2(x)$$
$$\vdots$$
$$a(x)y'' + b(x)y' + c(x)y = d_N(x)$$

Then find a particular solution y_{pi}

A particular solution to the original equation is given by:

$$\boxed{y_p(x) = y_{p1}(x) + y_{p2}(x) + \ldots + y_{pN}(x) = \sum_{i=1}^{i=N} y_{pi}(x)}$$

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

If the nonhomogeneous term $d(x)$ consist of several terms:

$$d(x) = d_1(x) + d_2(x) + \ldots + d_N(x) = \sum_{i=1}^{i=N} d_i(x)$$

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$$\vdots$$
$$a(x)y'' + b(x)y' + c(x)y = d_N(x)$$

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2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

1. Find a particular solution to the equation

$$y'' - 3y' - 4y = 3e^{2x} - 8e^{-x}$$