# Numerical Analysis

**Course:- B.Sc. III**

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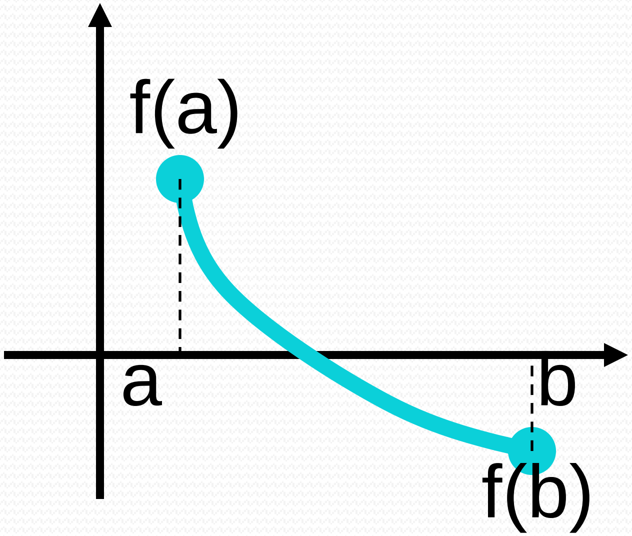
# OUTLINES

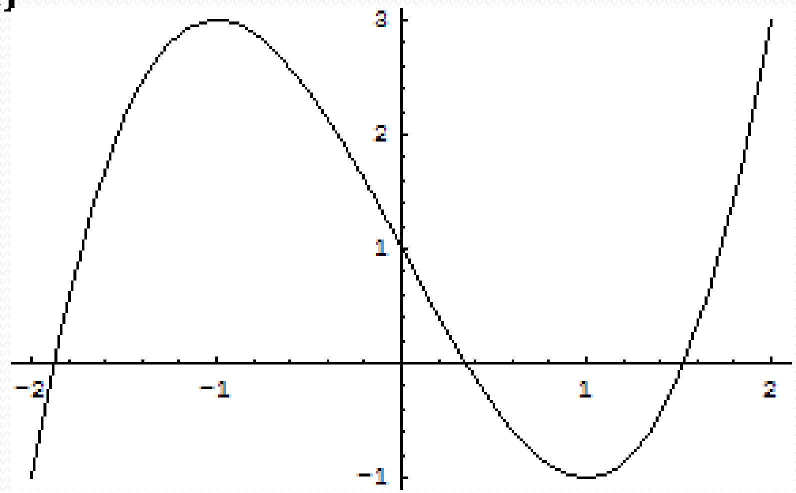
## Numerical Methods

- Bisection Method
- Regula-falsi Methods
- Newton-Raphson Method
- Newton's Forward and Backward Interpolation formula
- Gauss Forward and Backward,
- Bessel's Interpolation formula
- Sterling's and Evertt's Interpolation formula.
- Numerical Differentiations.
- Numerical Integration using Trapezoidal, Simpson 1/3 and Simpson 3/8 rule.

# Intermediate Value Theorem

If a function f(x) is continuous on some interval [a,b] and f(a) and f(b) have different signs then the equation f(x)=0 has at least one real root (zero) or an odd number of real roots in the interval [a,b].



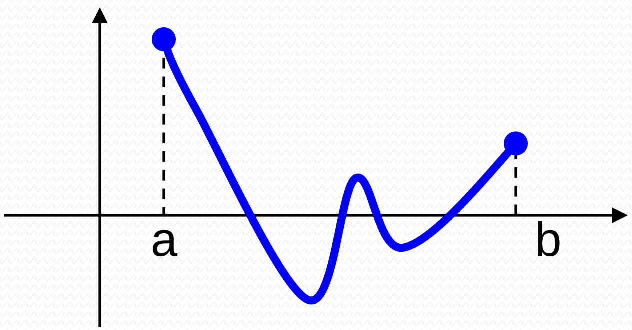



# Bisection Method

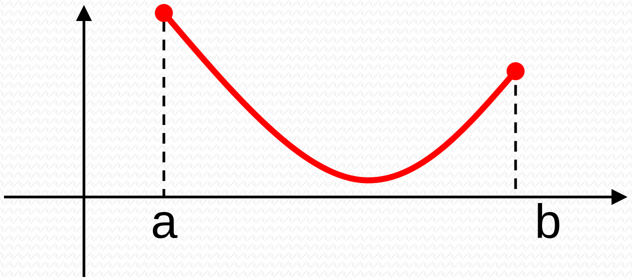
- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

# Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].
- Bisection method can not be used in these cases.




The function has four real zeros




The function has no real zeros

# Bisection Method

**Assumptions:**

Given an interval [a,b]

f(x) is continuous on [a,b]

f(a) and f(b) have opposite signs.

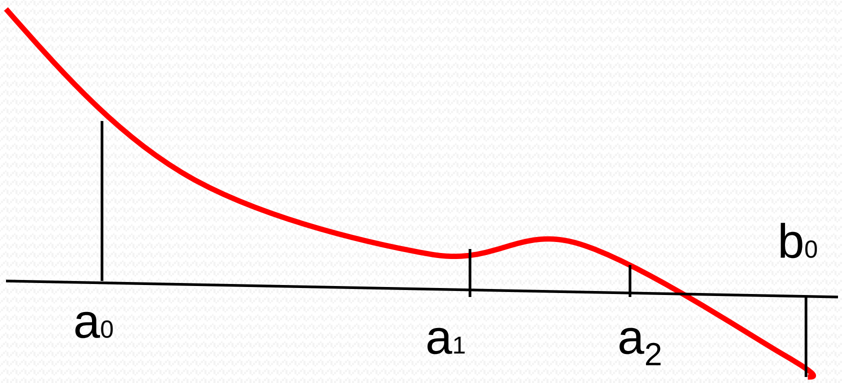
These assumptions ensures the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

# Stopping Criteria

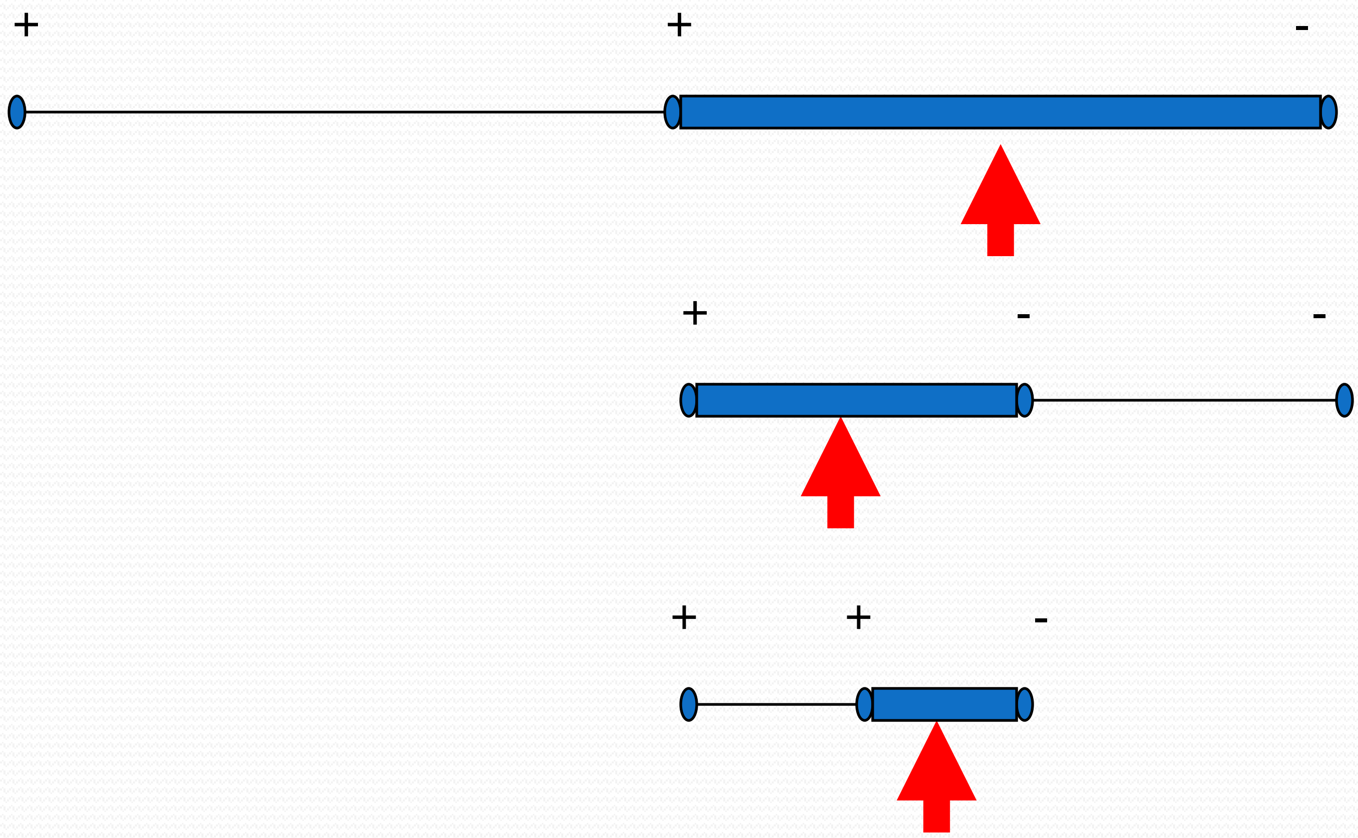
Two common stopping criteria

1. Stop after a fixed number of iterations.
2. Stop when the two approximate values $x_n$ and $x_{n+1}$ are equal.

# Bisection Method



# Example



# Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,2]?

**Answer:**

$f(x)$ is continuous on $[0,2]$

and $f(0) * f(2) = (1)(3) = 3 > 0$

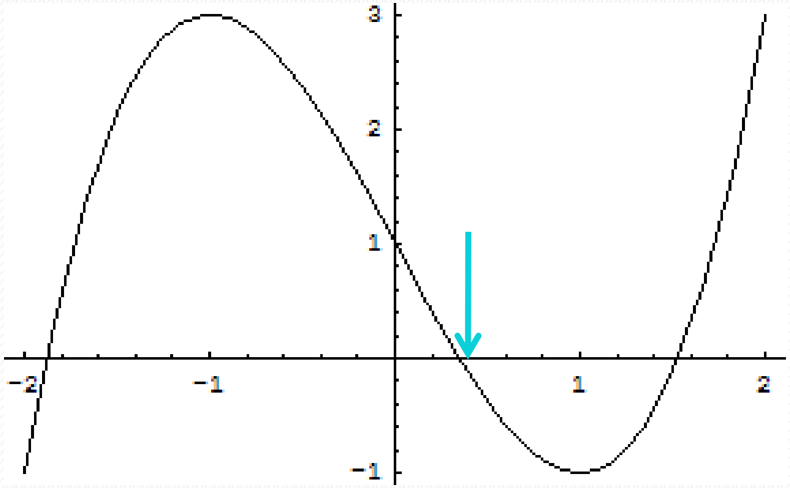
$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

# Example:

Can you use Bisection method to find a zero of

$f(x) = x^3 - 3x + 1$ in the interval $[0,1]$?




**Answer:**

$f(x)$ is continuous on $[0,1]$

$\mathrm{f}(0) * \mathrm{f}(1) = (1)(-1) = -1 < 0$

Assumptions are satisfied

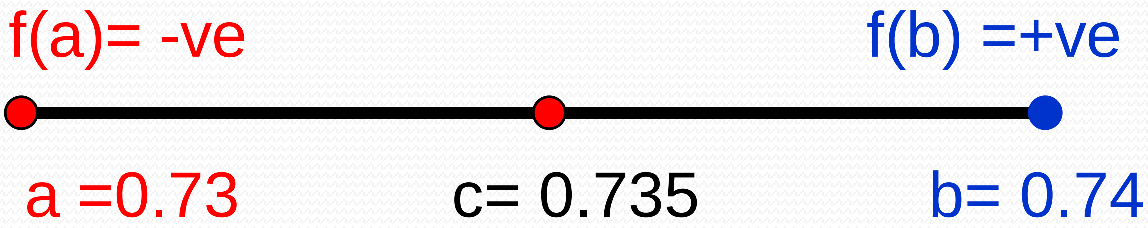
Bisection method can be used

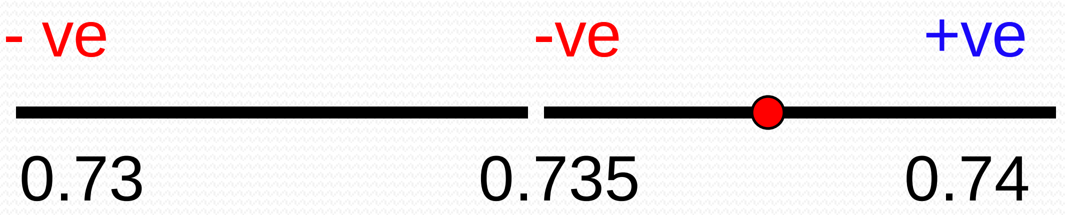
# Example

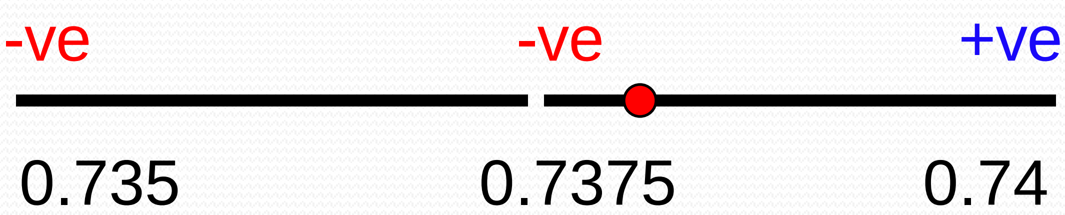
- Use Bisection method to find a root of the equation
  $x = \cos(x)$.
  (assume the initial interval [0.73,0.74])

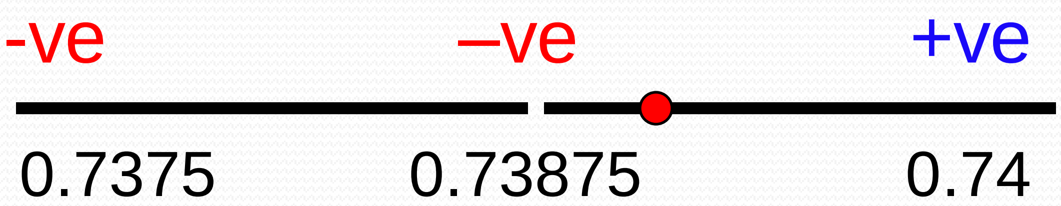
Question 1: What is $f(x)$ ?
Question 2: Are the assumptions satisfied ?

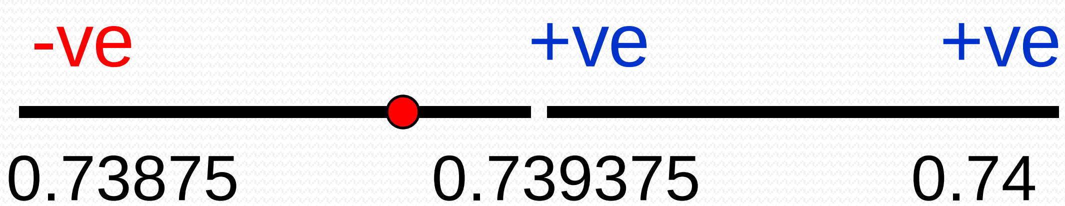
# Bisection Method
## Initial Interval




- ve
-ve
+ve
0.73
0.735
0.74


-ve
-ve
+ve
0.735
0.7375
0.74


-ve
–ve
+ve
0.7375
0.73875
0.74


-ve
+ve
+ve
0.73875
0.739375
0.74

# Summary

- Initial interval containing the root [0.73,0.74]
- After 8 iterations
  - Interval containing the root [0.7390625 ,0.73914]
  - Best estimate of the root is 0.73910.

# Bisection Method

**Advantages**

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- No knowledge of the derivative is needed
- The function does not have to be differentiable

**Disadvantage**

- Slow to converge
- Good intermediate approximations may be discarded

# Regula Falsi Method

- The convergce process in the bisection method is very slow.
- It depends only on the choice of end points of the interval [a,b].
- The function f(x) does not have any role in finding the point c (which is just  the mid-point of a and b).
- It is used only to decide the next smaller interval [a,c] or [c,b].

Consider the equation f(x)=o and let a and b be two values of x that f(a) and f(b) are of opposite signs. Also let a<b. the graph of y=f(x) will meet the x-axis at the same point between a and b, the equation chord joining the two points [ a, f(a)] and [b, f(b)] is

$$\frac{y-f(a)}{x-a}=\frac{f(b)-f(a)}{b-a}$$

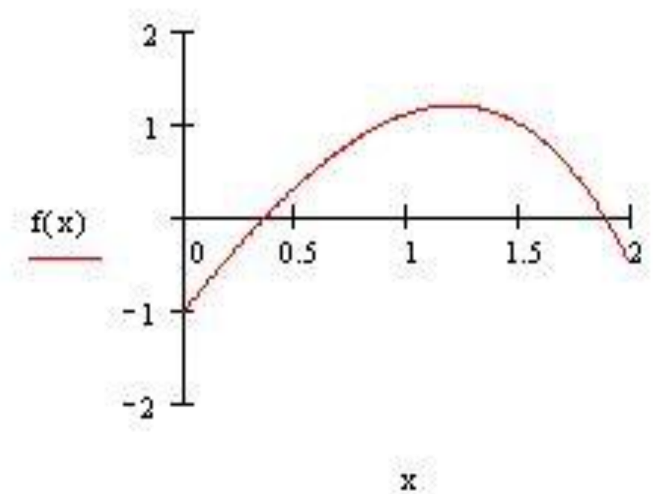
in the small interval (a, b) the graph of the function can be considered as a straight line. So that x-coordinate of the point of intersection of the chord joining [a, f(a)] and [b, f(b)] with the x-axis will give an approximate value of the root. So putting y=o.

$$\frac{f(a)}{x-a}=\frac{f(b)-f(a)}{b-a} \quad \rightarrow x=a-\frac{f(a)}{f(b)-f(a)}(b-a)$$

$$or \quad x=\frac{a\,f(b)-b\,f(a)}{f(b)-f(a)}$$

# Find a root of 3x + sin(x) - exp(x) = 0.

- The graph of this equation is given in the figure.
- From this it's clear that there is a root between 0 and 0.5 and also another root between 1.5 and 2.0.
- Now let us consider the function f (x) in the interval [0, 0.5] where f (0) * f (0.5) is less than zero and use the regula-falsi scheme to obtain the zero of f (x) = 0.




| Iteration No. | a | b | c | f(a) * f(c) |
|---|---|---|---|---|
| 1 | 0 | 0.5 | 0.376 | 1.38 (+ve) |
| 2 | 0.376 | 0.5 | 0.36 | -0.102 (-ve) |
| 3 | 0.376 | 0.36 | **0.36** | -0.085 (-ve) |

# Newton-Raphson Method
(also known as Newton's Method)

Given an initial guess of the root $x_0$ , Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.

## Assumptions:

- $f(x)$ is continuous and first derivative is known
- An initial guess $x_0$ such that $f'(x_0) \neq 0$ is given

# Derivation of Newton's Method

Let $x_0$ be an approximate root of equation $f(x) = 0.$ *If* $x_1 = x_0 + \mathrm{h}$ be the exact root, then $\mathrm{f}(x_1) = 0$, then $\mathrm{f}(x_0 + \mathrm{h}) = 0$

The Taylor's expansion -

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \ldots.. = 0$$

since $h$ is small, neglecting $h^2$ and higher power of h.

$$f(x_0) + hf'(x_0) = 0 \Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

which is known as the Newton-Raphson formula or Newton's iteration formula.

# Example

Find a zero of the function $f(x) = x^3 - 2x^2 + x - 3$.

Solution : $x_0 = 4,\ f'(x) = 3x^2 - 4x + 1$

Iteration 1 : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 4 - \dfrac{33}{33} = 3$

Iteration 2 : $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3 - \dfrac{9}{16} = 2.4375$

Iteration 3 : $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.4375 - \dfrac{2.0369}{9.0742} = 2.2130$

# Summary

| | |
|---|---|
| Bisection, Regula Falsi Method | Reliable, Slow<br>One function evaluation per iteration<br>Needs an interval [a,b] containing the root,  f(a)  f(b)<0<br>No knowledge of derivative is needed |
| Newton Raphson Method | Fast  (if near the root) but may diverge<br>Two  function evaluation per iteration<br>Needs derivative  and an initial guess $x_0$,  $f'(x_0)$  is nonzero |